

# RATIONAL ANALOGS OF PROJECTIVE PLANES

ZHIXU SU

**ABSTRACT.** In this paper, we study the existence of high dimensional closed smooth manifolds whose rational homotopy type resembles that of a projective plane. Applying rational surgery, the problem can be reduced to finding possible Pontryagin numbers satisfying the Hirzebruch signature formula and a set of congruence relations, which turns out to be equivalent to finding solutions to a system of Diophantine equations.

## 1. INTRODUCTION

There are four kinds of projective planes, the well-known real, complex, quaternionic and octonionic projective planes. There does not exist any higher dimensional closed manifold having the topological structure of a projective plane. More precisely, for  $n > 8$ , there does not exist any simply-connected  $2n$ -dimensional closed manifold  $M$  such that

$$H^*(M; \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, n, 2n; \\ 0 & \text{otherwise} \end{cases}$$

This fact is a consequence of the well-known Hopf Invariant One Theorem. Suppose there is such a manifold  $M^{2n}$  for  $n > 8$ , then there must exist a Morse function with minimal number of critical points which gives a CW complex  $X = e^0 \cup e^n \cup_\phi e^{2n}$  that is homotopy equivalent to  $M$ . This indicates the existence of a Hopf invariant 1 attaching map  $\phi : S^{2n-1} \rightarrow S^n$ . But the only such maps are the Hopf fibrations  $S^{2k-1} \rightarrow S^k$  for  $k = 1, 2, 4, 8$ .

Ignoring torsion, we ask if any rational analogs of projective plane exist in higher dimension. This paper proves the following result

**Theorem 1.1.** *After dimension 2, 4, 8, and 16, which are the dimension of  $\mathbb{RP}^2$ ,  $\mathbb{CP}^2$ ,  $\mathbb{HP}^2$  and  $\mathbb{OP}^2$  respectively, the smallest next dimension where a rational analog of projective plane exists is 32. i.e. there exist 32-dimensional simply-connected smooth closed manifolds  $M$  such that*

$$H^*(M; \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0, 16, 32; \\ 0 & \text{otherwise} \end{cases}$$

*and such manifolds fall into infinitely many homeomorphism types.*

From the desired intersection form, it is immediate that such a manifold only exists in dimension  $4k$ . We will firstly show that there is no such manifold in dimension  $4k$  where  $k$  is odd. Then as we study the candidate dimensions, 24 turns out to give a negative answer. In dimension 32, we can

find infinitely many homeomorphism types of rational projective planes in terms of their Pontryagin numbers.

The main tool to prove the results is the rational surgery Realization Theorem, which was firstly introduced by Barge in [Barge, Theorem 1] and [Sullivan]; equivalent statements can be found in [TW]. The theorem gives a constructive answer to the existence question by finding pairings of  $4i$ -dimensional cohomology classes and choice of fundamental class that act like Pontryagin numbers. In section 2, we will state the rational surgery Realization Theorem, phrase it in a form that is ready for application to our problem. To make the theorem more transparent, an outline of the proof will be given. In section 3, we will prove Theorem 1.1.

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## 2. RATIONAL SURGERY

Given a rational homotopy type, a natural question is whether there exists a closed manifold realizing the rational homotopy data. Compared to its integral version, the existence question in rational setting has a more explicit solution. Philosophically, this is due to the much simpler rational homotopy groups of spheres. Initiated by Barge [Barge] and Sullivan [Sullivan], rational surgery constructs closed manifold that is rational homotopy equivalent to a proposed  $\mathbb{Q}$ -local space  $X^n$ , which is a CW complex whose homotopy groups are  $\mathbb{Q}$ -vector spaces. Apparently, to get any positive answer, it is necessary to start with a local space  $X$  that satisfies Poincaré duality in rational coefficients. The ingredients for constructing a realizing manifold include choices of cohomology classes in  $H^{4i}(X; \mathbb{Q})$ , which play the role of Pontryagin classes, and correspondingly, a suitable choice of fundamental class in  $H_n(X; \mathbb{Q}) \cong H_n(X; \mathbb{Z}) \cong \mathbb{Q}$ .

**Theorem 2.1** ([Barge], [Sullivan]). *Let  $X$  be an  $n = 4k$ -dimensional simply-connected,  $\mathbb{Q}$ -local,  $\mathbb{Q}$ -Poincaré complex. There exists a simply-connected smooth closed  $4k$ -dimensional manifold  $M$ , and a  $\mathbb{Q}$ -homotopy equivalence  $f : M \rightarrow X$  if and only if: There exists cohomology classes  $p_i \in H^{4i}(X; \mathbb{Q})$ ,  $1 \leq i \leq k$ , and a fundamental class  $\mu \in H_{4k}(X; \mathbb{Q}) \cong \mathbb{Q}$  such that*

- (i)  $\langle L_k(p_1, \dots, p_k), \mu \rangle = \sigma(X)$
- (ii) *The intersection form  $\lambda : H^{2k}(X; \mathbb{Q}) \times H^{2k}(X; \mathbb{Q}) \rightarrow \mathbb{Q}$  defined as  $\langle \cdot \cup \cdot, \mu \rangle$  is isomorphic to  $p\langle 1 \rangle \oplus q\langle -1 \rangle$*
- (iii) *For each partition  $I$  of  $k$ , the pairing  $\langle p_I, \mu \rangle$  agrees with the  $I$ 's Pontryagin number of a closed smooth manifold, i.e., there exists a closed smooth  $4k$ -dimensional manifold  $N$  such that*

$$\langle p_I(\tau_N), [N] \rangle = \langle p_I, \mu \rangle$$

*for all partitions  $I$  of  $k$ .*

*Proof.* We will claim that for any choice of cohomology classes  $p_i$ 's together with a fundamental class  $\mu$  satisfying the above three conditions, surgery can be applied to construct a  $\mathbb{Q}$ -homotopy equivalence  $f : M \rightarrow X$  such that  $f_*[M] = \mu$  and  $f^*p_i = p_i(\tau_M)$ . Condition (iii) guarantees a degree 1 normal map from a candidate manifold  $M$  to  $X$  so that the fundamental class of  $M$  is sent to the chosen class  $\mu$ . Condition (i) and (ii) ensure the vanishing surgery obstruction.

Consider any choice of cohomology classes  $p : X \xrightarrow{(p_1, \dots, p_k)} \prod K(\mathbb{Q}, 4i) \simeq BSO_{(0)}$ . For  $m \gg n$ , let  $\bar{p} : BSO(m) \xrightarrow{(\bar{p}_1, \dots, \bar{p}_i, \dots)} \prod K(\mathbb{Q}, 4i)$  be the total Pontryagin class of the Whitney sum inverse bundle of the universal plane bundle  $\gamma^m$  over  $BSO(m)$ . Let  $PB$  be the homotopy pull-back space of  $p$  and  $\bar{p}$ , and  $\xi^m$  the pullback bundle of  $\gamma^m$  over  $PB$ . We have constructed the right two columns of the following diagram. Note that  $\bar{p}$  and the projection map  $pr_1$  are localization maps by construction.

$$\begin{array}{ccccc}
 \nu_M & \longrightarrow & \xi & \longrightarrow & \gamma^m \\
 \downarrow & & \downarrow & & \downarrow \\
 M & \xrightarrow{g} & PB & \xrightarrow{pr_2} & BSO(m) \\
 & \searrow f & \downarrow pr_1 & & \downarrow \bar{p} \\
 & & X & \xrightarrow{p} & \prod K(\mathbb{Q}, 4i)
 \end{array}$$

For any homotopy class  $\alpha \in \pi_{n+m}(T\xi^m)$ , the corresponding map  $g : S^{m+n} \rightarrow T\xi^m$  yields a candidate manifold  $M = \alpha^{-1}(PB)$  by Thom-Pontryagin construction. Moreover,  $g|_M : M \rightarrow PB$  is covered by a bundle map from the normal bundle of  $M$  to  $\xi$ . Chasing the diagram, one can check that the input classes  $p_i$ 's are pulled back to the Pontryagin classes  $p_i(\tau_M)$  through the composition map  $f := pr_1 \circ g : M \rightarrow X$ .

To construct a degree 1 normal map so that  $f_*[M] = \mu$ , we need a particular class  $\alpha \in \pi_{n+m}(T\xi^m)$  that maps to  $\mu$  under the composition of Hurewicz map, Thom isomorphism, and the projection  $pr_{1*} : H_n(PB; \mathbb{Z}) \rightarrow H_n(X; \mathbb{Z})$ , which is shown the following diagram

$$\begin{array}{ccccc}
 \alpha \in \pi_{n+m}(T\xi^m) & \xrightarrow{T_{pr_2*}} & \pi_{n+m}(T\gamma^m) \ni \beta & & \\
 \downarrow T_{pr_1*} & \searrow & \downarrow T_{\bar{p}*} & & \\
 & H_n(PB) \xrightarrow{pr_{2*}} H_n(BSO(m)) & & & \\
 & \downarrow pr_{1*} & \downarrow \bar{p}_* & & \\
 & \mu \in H_n(X) \xrightarrow{p_*} H_n(BSO(m)_{(0)}) & & & \\
 \nearrow \cong & & \nwarrow \cong & & \\
 c_X \in \pi_{n+m}(T\tilde{\nu}_X) & \xrightarrow{T_{p*}} & \pi_{n+m}(T\gamma_{(0)}^m) & & 
 \end{array}$$

In the lower right corner of the diagram,  $T\gamma_{(0)}^m$  is the Thom space associated to the rational spherical fibration  $S_{(0)}^{m-1} \rightarrow S\gamma_{(0)}^m \rightarrow BSO(m)_{(0)}$ , which is the localization of the sphere bundle  $S^{m-1} \rightarrow S\gamma^m \rightarrow BSO(m)$ . The Hurewicz Thom map

$$\pi_{n+m}(T\gamma_{(0)}^m) \rightarrow H_{n+m}(T\gamma_{(0)}^m; \mathbb{Z}) \rightarrow H_n(BSO(m)_{(0)}; \mathbb{Z})$$

is an isomorphism since both the Thom space and the base space are  $\mathbb{Q}$ -local and the Hurewicz map is a rational isomorphism for  $m \gg n$  ([MS, Theorem 18.3]). In the lower left corner, the rational spherical fibration  $\tilde{\nu}_X = p^*(S\gamma_{(0)}^m)$  and the associated Thom space  $T\tilde{\nu}_X$  are  $\mathbb{Q}$ -local, the Hurewicz Thom map  $\pi_{n+m}(T\tilde{\nu}_X) \rightarrow H_n(X; \mathbb{Z})$  is also an isomorphism. Then for any fundamental class  $\mu$ , there is a class  $c_X \in \pi_{n+m}(T\tilde{\nu}_X)$  mapping to  $\mu$ . Moreover, it can be shown that the outer square of Thom space is a homotopy cartesian square (See [TW, Lemma 6.1] or [Su, Lemma 3.2.3] for more details). All these together imply that if there exists a class  $\beta \in \pi_{n+m}(T\gamma^m)$  in the upper right corner mapping to  $p_*\mu \in H_n(BSO(m)_{(0)})$ ,  $\beta$  and  $c_X$  would guarantee the existence of a desired class  $\alpha$  that maps to  $\mu$ .

Note that the Hurewicz Thom map in the upper right corner can be viewed as  $\nu : \pi_{n+m}(T\gamma^m) \cong \Omega_n^{SO} \rightarrow H_n(BSO; \mathbb{Q})$  where  $\nu(M) = \nu_{M*}[M]$  and  $\nu_M$  is the classifying map of the normal bundle of a manifold  $M$ . Then there is a  $\beta$  mapping to  $p_*\mu$  if and only if  $\bar{p}_*^{-1}(p_*\mu)$  lies in the image of such map  $\nu$ .

If the input classes  $p_i$ 's and  $\mu$  together satisfy condition (iii), i.e., there exists a closed manifold  $N$  such that  $\langle p_I(\tau_N), [N] \rangle = \langle p_I, \mu \rangle$ , chasing the diagram, we have

$$\langle p_I(\tau_N), [N] \rangle = \langle p_I, \mu \rangle = \langle \bar{p}_I, \bar{p}_*^{-1}(p_*\mu) \rangle$$

Since for any bundle  $\eta$ , the Pontryagin numbers of the inverse bundle  $p_I(\bar{\eta})$  can be written as linear combination of  $p_I(\eta)$ 's, the above identity implies that  $\langle p_I(\nu_N), [N] \rangle = \langle p_I(\gamma^m), \bar{p}_*^{-1}(p_*\mu) \rangle$ , which is equivalent to saying that  $\bar{p}_*^{-1}(p_*\mu)$  is the image of manifold  $N$  under the homomorphism  $\nu : \Omega_n^{SO} \rightarrow H_n(BSO; \mathbb{Q})$ . This implies that  $\pi_{n+m}(T\gamma^m)$  possesses the desired class  $\beta$  and thus ensures the existence of  $\alpha$ , which finishes the proof that condition (iii) guarantees a degree 1 normal map such that  $f_*[M] = \mu$ .

Now surgery can be applied to alter the normal map to a rational homotopy equivalence if and only if the map has a vanishing surgery obstruction, which lives in the  $L$  group

$$L_n(\mathbb{Q}) \cong \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \oplus \bigoplus_{\infty} \mathbb{Z}_4$$

See [MH]. The obstruction vanishes in its  $\mathbb{Z}$  summand if and only if the signature  $\sigma(M) = \sigma(X)$ , which is equivalent to condition (i) since

$$\begin{aligned} \langle L_k(p_1, \dots, p_k), \mu \rangle &= \langle L_k(p_1, \dots, p_k), f_*[M] \rangle \\ &= \langle L_k(f^*p_1, \dots, f^*p_k), [M] \rangle \\ &= \langle L_k(p_1(\tau_M), \dots, p_k(\tau_M)), [M] \rangle \\ &= \sigma(M) \end{aligned}$$

Condition (ii) requires the rational intersection form of  $X$  to be a direct sum of  $\langle 1 \rangle$ 's and  $\langle -1 \rangle$ 's, which guarantees the obstruction vanishes in its  $\mathbb{Z}_2$  and  $\mathbb{Z}_4$  summands in  $L_n(\mathbb{Q})$ . This finished the outline of the proof of Theorem 2.1.  $\square$

**Remark 2.2.** One can also ask the existence of any closed topological or piecewise linear manifold realizing the rational homotopy type of projective planes. The Realization Theorem 2.1 still works for the *PL* or *TOP* category by changing the word “smooth” in condition (iii) to PL or topological.

### 3. RATIONAL PROJECTIVE PLANES

In this section, we study the existence dimension of rational projective planes. Recall that we ask the smallest dimension  $4k$  ( $> 16$ ) where a simply-connected closed smooth manifold  $M$  such that

$$H^*(M; \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0, 2k, 4k; \\ 0 & \text{otherwise} \end{cases}$$

exists. Equivalently, we look for the existence dimension of simply-connected closed smooth manifolds that are rational homotopy equivalent to a  $4k$ -dimensional  $\mathbb{Q}$ -local,  $\mathbb{Q}$ -Poincaré complex  $X$  where

$$H^*(X; \mathbb{Z}) \cong H^*(X; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & * = 0, 2k, 4k; \\ 0 & \text{otherwise} \end{cases}$$

the problem then can be studied using the rational surgery realization Theorem 2.1.

**3.1. The target  $\mathbb{Q}$ -local space.** Firstly, we construct  $X$  from a Postnikov tower of rational principal fibration. Let  $X \rightarrow K(\mathbb{Q}, 2k)$  be the principal fibration with fiber  $K(\mathbb{Q}, 6k-1)$  and  $k$ -invariant  $\iota_{2k}^3$

$$\begin{array}{ccc} K(\mathbb{Q}, 6k-1) & \longrightarrow & K(\mathbb{Q}, 6k-1) \\ \downarrow & & \downarrow \\ X & \longrightarrow & * \\ \downarrow & & \downarrow \\ K(\mathbb{Q}, 2k) & \xrightarrow{\iota_{2k}^3} & K(\mathbb{Q}, 6k) \end{array}$$

Computing the spectral sequence, it is easy to check that  $X$  has the desired rational cohomology ring  $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x]/\langle x^3 \rangle$  with  $|x| = 2k$ . Notice that the signature  $\sigma(X) = \pm 1$  by our construction.

Now let's plugin  $X$  into the Realization Theorem 2.1. Note that since  $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x]/\langle x^3 \rangle$ , the input classes  $p_i \in H^{4i}(X; \mathbb{Q})$  is zero for all  $i$  except  $p_{\frac{k}{2}}$  and  $p_k$ , then our realization problem of rational projective plane is equivalent to asking the following:

**3.2. Equivalent question.** *For  $k > 4$ , can we find a choice of cohomology classes*

$$p_{\frac{k}{2}} \in H^{2k}(X; \mathbb{Q}), \quad \text{and} \quad p_k \in H^{4k}(X; \mathbb{Q})$$

*together with a fundamental class*

$$\mu \in H_{4k}(X; \mathbb{Z}) \cong \mathbb{Q}$$

*such that*

- (i)  $\langle L_k(0, \dots, 0, p_{\frac{k}{2}}, 0, \dots, 0, p_k), \mu \rangle = \pm 1$
- (ii) *The intersection form on  $H^{2k}(X; \mathbb{Q})$  with respect to  $\mu$  is isomorphic to a direct sum of  $\langle 1 \rangle$ 's and  $\langle -1 \rangle$ 's.*
- (iii) *There exists a closed  $4k$ -dimensional manifold  $N$  such that*

$$\langle p_I(\tau_N), [N] \rangle = \langle p_I, \mu \rangle$$

*for all partitions  $I$  of  $k$ .*

In the signature condition, the coefficient of  $p_k$  in the  $L$ -polynomial  $L_k(p_1, \dots, p_k)$  is

$$(3.2.1) \quad s_k = \frac{2^{2k}(2^{2k-1} - 1)B_{2k}}{(2k)!}$$

where  $B_{2k}$  is the  $2k$ -th Bernoulli number [MS]. As mentioned in [A], the coefficient of  $p_{\frac{k}{2}}^2$  can be computed as

$$(3.2.2) \quad \begin{aligned} s_{\frac{k}{2}, \frac{k}{2}} &= \frac{1}{2}(s_{\frac{k}{2}}^2 - s_k) \\ &= \frac{1}{2} \left( \left( \frac{2^k(2^{k-1} - 1)|B_k|}{k!} \right)^2 - \frac{2^{2k}(2^{2k-1} - 1)|B_{2k}|}{(2k)!} \right) \end{aligned}$$

Then condition (i) says

$$s_{\frac{k}{2}, \frac{k}{2}} \langle p_{\frac{k}{2}}^2, \mu \rangle + s_k \langle p_k, \mu \rangle = \pm 1$$

From condition (i) and (iii), we can narrow down the candidate dimensions to  $4k$  with  $k$  even.

**Lemma 3.1.** *There does not exist any rational projective plane in dimension  $4k$  when  $k$  is odd.*

*Proof.* When  $k$  is odd, the input Pontryagin class  $p_i$  is nonzero only when  $i = k$ . Then condition (i) requires:

$$\langle L_k(0, \dots, 0, p_k), \mu \rangle = s_k \langle p_k, \mu \rangle = \pm 1$$

where the numerator in the irreducible form of  $s_k$  is an integer not equal to  $\pm 1$ . On the other hand, condition (iii) requires  $\langle p_k, \mu \rangle$  to be Pontryagin number of a smooth closed manifold, which must be an integer. This is impossible if  $s_k \langle p_k, \mu \rangle = \pm 1$  but the numerator of  $s_k$  is not  $\pm 1$ . Thus condition (i) and (iii) can never be both satisfied in dimension  $n = 4k$  with  $k$  odd.  $\square$

**3.3. Dimension 24.** Lemma 3.1 indicates that  $n = 24$  is the next candidate. It turns out the signature formula can never be satisfied in this dimension.

**Lemma 3.2.** *There does not exist any rational projective plane in dimension 24.*

*Proof.* Condition (i) requires existence of cohomology classes  $p_3 \in H^{12}(X; \mathbb{Q}) \cong \mathbb{Q}$ ,  $p_6 \in H^{24}(X; \mathbb{Q}) \cong \mathbb{Q}$  and a choice of fundamental class  $\mu \in H_{24}(X; \mathbb{Z}) \cong \mathbb{Q}$  such that

$$s_{3,3} \langle p_3^2, \mu \rangle + s_6 \langle p_6, \mu \rangle = \pm 1$$

Let  $\alpha$  be any nonzero class in  $H^{12}(X, \mathbb{Q}) \cong \mathbb{Q}$ , one can write

$$p_3 = a\alpha, \quad p_3^2 = a^2\alpha^2$$

and

$$p_6 = b\alpha^2$$

for some nonzero rational number  $a$  and  $b$ . Correspondingly, let  $[X] \in H_{24}(X, \mathbb{Z}) \cong \mathbb{Q}$  be the fundamental class such that  $\langle \alpha \cup \alpha, [X] \rangle = 1$ .

In order to have a rational intersection form isomorphic to a direct sum of  $\langle 1 \rangle$ 's and  $\langle -1 \rangle$ 's, we need to choose a fundamental class  $\mu$  such that  $\mu = \pm r^2[X]$  for some nonzero rational number  $r$ .

Condition (iii) requires the pairings  $\langle p_3^2, \mu \rangle$  and  $\langle p_6, \mu \rangle$  to be integers. So we may let  $x$  and  $y$  be the integers such that  $x^2 = a^2 r^2$ ,  $y = b r^2$ , then

$$\langle p_3^2, \mu \rangle = \pm x^2, \quad \langle p_6, \mu \rangle = \pm y$$

Altogether, condition (i),(ii) and the integrality part of condition (iii) require the existence of integers  $x$  and  $y$  such that:

$$(3.3.1) \quad s_{3,3} x^2 + s_6 y = \pm 1$$

where the coefficients can be computed using formula (3.2.1), (3.2.2) to be

$$s_{3,3} = \frac{40247}{638512875}, \quad s_6 = -\frac{2828954}{638512875}$$

To solve the Diophantine equation (3.3.1), one can either compute by hand using quadratic reciprocity or refer to a mathematical software such as Mathematica. Equation (3.3.1) turns out to have no integer solution.  $\square$

In order to continue the analysis on the following candidate dimensions, we need to give condition (iii) an explicit interpretation.

**3.4. Congruence relations among Pontryagin numbers.** Condition (iii) requires the set of pairings  $\langle p_I, \mu \rangle$  to be Pontryagin numbers of any closed smooth manifold. These integers form a sublattice in  $\mathbb{Z}^{p(n)}$  which can be classified by a group of congruence relations. The following Hattori-Stong Theorem says that the Riemann-Roch Theorem and the integrality of Pontryagin numbers completely determine all the relations among the Pontryagin numbers of smooth closed manifolds.

**Theorem 3.3.** [St2, Theorem 3] *The image of the homomorphism*

$$\tau : \Omega_*^{SO}/\text{tor} \rightarrow H_*(BSO; \mathbb{Q})$$

*is a lattice consisting exactly the elements  $x \in H_*(BSO; \mathbb{Q})$  such that:*

$$(3.4.1) \quad \begin{cases} \langle \mathbb{Z}[e_1, e_2, \dots] \cdot L, x \rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle \mathbb{Z}[p_1(\gamma), p_2(\gamma), \dots], x \rangle \in \mathbb{Z} \end{cases}$$

where  $e_i \in H^*(BSO; \mathbb{Q})$  is the  $i$ -th elementary symmetric polynomial of the variables  $e^{x_j} + e^{-x_j} - 2$ , i.e.,

$$e_i = \sigma_i(e^{x_1} + e^{-x_1} - 2, e^{x_2} + e^{-x_2} - 2, \dots)$$

where the total Pontryagin class is formally expressed as  $p(\gamma) = \prod(1 + x_j^2)$ .

As we expand  $e^{x_j} + e^{-x_j} - 2$  as a power series of  $x_j^2$ , the classes  $e_i$  are symmetric polynomials of the variables  $x_j^2$ . Since any symmetric polynomial can be expressed in terms of elementary symmetric polynomials, and the Pontryagin class  $p_i(\gamma)$  is exactly the  $i$ -th elementary symmetric polynomial of the variables  $x_j^2$ , we can express each  $e_i$  class in terms of the Pontryagin classes  $p_1(\gamma), p_2(\gamma), \dots$ . Therefore the relations (3.4.1) in the theorem provide a set of integrality conditions on the numbers

$$\langle p_I(\gamma), x \rangle = \langle p_I(\gamma), \tau_*[N] \rangle = \langle p_I(\tau_N), [N] \rangle$$

for any smooth manifold  $N \in \Omega_*^{SO}$ . These integrality conditions determine all the possible Pontryagin numbers of a closed smooth manifold. Then condition (iii) in the Realization Theorem 2.1 is equivalent to asking for input  $p_i$  and  $\mu$  from the local space  $X$  such that

$$\begin{cases} \langle \mathbb{Z}[e_1, e_2, \dots] \cdot L, \mu \rangle \in \mathbb{Z}[\frac{1}{2}] \\ \langle \mathbb{Z}[p_1, p_2, \dots], \mu \rangle \in \mathbb{Z} \end{cases}$$

In our case, we may express the  $e_i$  classes solely in terms of  $p_{\frac{k}{2}}$  and  $p_k$  by assuming that all the other Pontryagin classes are zero. The following example in dimension 16 illustrates how such expressions can be found in higher dimensions.

**Example 3.4.** Suppose we want to find the explicit congruence relations in dimension 16. The first thing we need to do is to express the 16-dimensional summand of  $\mathbb{Z}[e_1, e_2, \dots] \cdot L$  in terms of  $p_i$ . Since  $e_i$  consists of classes of dimension no less than  $4i$ , the 16-dimensional classes live in

$$(3.4.2) \quad (\mathbb{Z} \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_1^2 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_1e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_2^2 \oplus \mathbb{Z}e_1e_3 \oplus \mathbb{Z}e_4) \cdot L$$

As we assume the input class  $p_i = 0$  except  $p_2$  and  $p_4$ , the total  $L$ -class is

$$L = 1 + s_2p_2 + s_{2,2}p_2^2 + s_4p_4 = 1 - \frac{1}{45}p_2 - \frac{19}{14175}p_2^2 + \frac{381}{14175}p_4$$

each of the  $e_i$  class can be written as a linear combination of  $p_2, p_2^2$  and  $p_4$ . Take  $e_2$  for example, we firstly expand  $e^{x_j} + e^{-x_j} - 2$  as a power series

$$e^{x_j} + e^{-x_j} - 2 = x^2 + \frac{x^4}{12} + \frac{x^6}{360} + \frac{x^8}{20160} + O(x^9)$$

Then the symmetric polynomial

$$\begin{aligned} e_2 &= \sigma_2(e^{x_1} + e^{-x_1} - 2, e^{x_2} + e^{-x_2} - 2, \dots) \\ &= \sum_{j,k} (e^{x_j} + e^{-x_j} - 2)(e^{x_k} + e^{-x_k} - 2) \\ &= \sum_{j,k} \left( x_j^2 + \frac{x_j^4}{12} + \frac{x_j^6}{360} + \frac{x_j^8}{20160} + O(x_j^9) \right) \left( x_k^2 + \frac{x_k^4}{12} + \frac{x_k^6}{360} + \frac{x_k^8}{20160} + O(x_k^9) \right) \\ &= \sum_{j,k} \left( x_j^2 x_k^2 + \frac{x_j^4 x_k^4}{144} + \frac{x_j^2 x_k^6}{360} + \frac{x_j^6 x_k^2}{360} \right) + \text{terms of degree other than 8 and 16} \\ &= p_2 + \cancel{\frac{p_1^2 p_2^2}{360}} + \frac{p_2^2}{720} - \cancel{\frac{p_1 p_3}{60}} + \frac{p_4}{40} + \text{terms of degree other than 8 and 16} \end{aligned}$$

where the simplification in the last step can be done with the help of Mathematica. Then the condition from the summand  $\mathbb{Z}e_2 \cdot L$  is

$$\begin{aligned} \langle \mathbb{Z}e_2 \cdot L, \mu \rangle &= \mathbb{Z} \langle (p_2 + \frac{p_2^2}{720} + \frac{p_4}{40}) \cdot (1 - \frac{1}{45}p_2 - \frac{19}{14175}p_2^2 + \frac{381}{14175}p_4), \mu \rangle \\ &= \mathbb{Z} \langle -\frac{p_2^2}{48} + \frac{p_4}{40}, \mu \rangle \in \mathbb{Z}[\frac{1}{2}] \end{aligned}$$

Since we also require that  $\langle p_2^2, \mu \rangle, \langle p_4, \mu \rangle \in \mathbb{Z}$ , the condition is equivalent to the congruence relation

$$-5\langle p_2^2, \mu \rangle + 6\langle p_4, \mu \rangle \equiv 0 \pmod{15}$$

To find the complete set of congruence relations in dimension 16, one does the same process for each of the summand in (3.4.2).

One may also use the approach in Remark 3.5 to find the explicit congruence relations in terms of the Pontryagin classes, but we will use the method shown in the above example in the following section.

**3.5. Dimension 32.** In this dimension, we ask the existence of a closed smooth manifold that is rational homotopy equivalent to a  $\mathbb{Q}$ -local space  $X$  where

$$H^*(X; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & * = 0, 16, 32; \\ 0 & \text{otherwise} \end{cases}$$

Applying the Realization Theorem 2.1, we look for cohomology classes  $p_4$  and  $p_8$  in  $H^*(X; \mathbb{Q})$ , together with a choice of fundamental class  $\mu \in H_{32}(X; \mathbb{Z})$ , such that condition (i), (ii) and (iii) are satisfied. We can convert the problem to solving a system of diophantine equations.

Similar to the analysis on dimension 24, condition (ii) and the integrality of Pontryagin numbers ensure that we may let

$$\langle p_3^2, \mu \rangle = \pm x^2, \quad \langle p_6, \mu \rangle = \pm y$$

where  $x$  and  $y$  are integers. The signature condition requires the existence of integers  $x$  and  $y$  such that:

$$(3.5.1) \quad s_{4,4}x^2 + s_8y = \pm 1$$

where the coefficients can be computed as

$$s_{4,4} = -\frac{444721}{162820783125}, \quad s_8 = \frac{118518239}{162820783125}$$

One can find infinitely many solutions to the Diophantine equation (3.5.1), then it remains to be checked if there is any solution satisfies the congruence relations from condition (iii) at the same time, i.e.,

$$(3.5.2) \quad \langle \mathbb{Z}[e_1, e_2, \dots] \cdot L, \mu \rangle \in \mathbb{Z}[\frac{1}{2}]$$

To make the integrality condition explicit in this dimension, we expand each basis class of  $\mathbb{Z}[e_1, e_2, \dots]$  as power series on  $p_4$  and  $p_8$ , since we only care about the cohomology classes in dimension 32, we throw away the higher degree classes in the representations

$$\left\{ \begin{array}{l} e_1 = -\frac{1}{5040}p_4 + \frac{1}{2615348736000}p_4^2 - \frac{1}{1307674368000}p_8 \\ e_2 = \frac{1}{40}p_4 + \frac{3119}{435891456000}p_4^2 + \frac{5461}{217945728000}p_8, \quad e_1e_1 = \frac{1}{25401600}p_4^2 \\ e_3 = -\frac{1}{3}p_4 + \frac{19}{39916800}p_4^2 - \frac{31}{2851200}p_8, \quad e_1e_2 = -\frac{1}{201600}p_4^2, \quad e_1^3 = 0 \\ e_4 = p_4 + \frac{1}{1209600}p_4^2 + \frac{457}{604800}p_8, \quad e_1e_3 = \frac{1}{15120}p_4^2, \quad e_2e_2 = \frac{1}{1600}p_4^2 \\ e_5 = -\frac{43}{2520}p_8, \quad e_1e_4 = -\frac{1}{5040}p_4^2, \quad e_2e_3 = -\frac{1}{120}p_4^2, \\ e_6 = \frac{29}{180}p_8, \quad e_2e_4 = \frac{1}{40}p_4^2, \quad e_3e_3 = \frac{1}{9}p_4^2, \quad e_1e_5 = 0 \\ e_7 = -\frac{2}{3}p_8, \quad e_3e_4 = -\frac{1}{3}p_4^2, \quad e_2e_5 = 0, \quad e_1e_6 = 0 \\ e_8 = p_8, \quad e_4e_4 = p_4^2, \quad e_3e_5 = 0, \quad e_2e_6 = 0, \quad e_1e_7 = 0 \end{array} \right.$$

Multiplying the nonzero basis class on  $e_i$  with the total  $L$  class

$$L = 1 + L_4 + L_8 = 1 + \frac{241}{14175}p_4 - \frac{444721}{162820783125}p_4^2 + \frac{118518239}{162820783125}p_8$$

we obtain a basis for  $\mathbb{Z}[e_1, e_2, \dots] \cdot L$  consisting of linear combinations of  $p_4^2$  and  $p_8$  in dimension 32.

$$\left\{ \begin{array}{l} 1 \cdot L = -\frac{444721}{162820783125}p_4^2 + \frac{118518239}{162820783125}p_8 \\ e_1 \cdot L = -\frac{1260361}{373621248000}p_4^2 - \frac{1}{1307674368000}p_8 \\ e_2 \cdot L = \frac{185276207}{435891456000}p_4^2 + \frac{5461}{217945728000}p_8, \quad e_1 e_1 \cdot L = \frac{1}{25401600}p_4^2 \\ e_3 \cdot L = -\frac{678599}{119750400}p_4^2 - \frac{31}{2851200}p_8, \quad e_1 e_2 \cdot L = -\frac{1}{201600}p_4^2 \\ e_4 \cdot L = \frac{61699}{3628800}p_4^2 + \frac{457}{604800}p_8, \quad e_1 e_3 \cdot L = \frac{1}{15120}p_4^2, \quad e_2 e_2 \cdot L = \frac{1}{1600}p_4^2 \\ e_5 \cdot L = -\frac{43}{2520}p_8, \quad e_1 e_4 \cdot L = -\frac{1}{5040}p_4^2, \quad e_2 e_3 \cdot L = -\frac{1}{120}p_4^2 \\ e_6 \cdot L = \frac{29}{180}p_8, \quad e_2 e_4 \cdot L = \frac{1}{40}p_4^2, \quad e_3 e_3 \cdot L = \frac{1}{9}p_4^2 \\ e_7 \cdot L = -\frac{2}{3}p_8, \quad e_3 e_4 \cdot L = -\frac{1}{3}p_4^2 \\ e_8 \cdot L = p_8, \quad e_4 e_4 \cdot L = p_4^2 \end{array} \right.$$

Then the integrality condition (3.5.2) holds if and only if each basis class satisfies the relation

$$(3.5.3) \quad \langle -, \mu \rangle \in \mathbb{Z}[\frac{1}{2}]$$

We have set up integers  $x$  and  $y$  so that  $\langle p_4^2, \mu \rangle = \pm x^2$  and  $\langle p_8, \mu \rangle = \pm y$ . As we simplify the coefficients and throw away the redundant relations, (3.5.3) is equivalent to the following set of congruence relations on integers  $x$  and  $y$

$$(3.5.4) \quad \left\{ \begin{array}{l} 162820783125 \mid -444721x^2 + 118518239y \\ 638512875 \mid 8822527x^2 + 2y \\ 212837625 \mid 185276207x^2 + 10922y \\ 467775 \mid 678599x^2 + 1302y \\ 14175 \mid 61699x^2 + 2742y \\ 99225 \mid x^2 \\ 315 \mid y \end{array} \right.$$

Combining these relations with the signature condition (3.5.1), we have converted condition (i), (ii) and (iii) to a system of Diophantine equations, the existence question is equivalent to finding integer solutions to the system. One can solve by hand or use Mathematica to check that the system consisting of (3.5.1) and (3.5.4) has infinitely many solutions. For example,

$$x = 1308536224920225, \quad y = 6425012065870154712076616250$$

is one pair of solutions.

Recall that in the rational surgery Realization Theorem 2.1, as we construct a  $\mathbb{Q}$ -homotopy equivalence  $f : M \rightarrow X$ , Pontryagin numbers of the

resulting manifold  $M$  are realized by the input pairings, i.e.

$$\langle p_I, \mu \rangle = \langle p_I(\tau_M), [M] \rangle$$

Therefore distinct solutions of integers  $x$  and  $y$  in dimension 32 correspond to distinct pairs of Pontryagin numbers of the realizing manifold in this dimension. Since Pontryagin numbers are homeomorphism invariants, we have found infinitely many homeomorphism types of smooth closed manifolds which are rational analogs of projective planes. This ends the proof of our main theorem 1.1.

**Remark 3.5.** There is another approach to compute the congruence relations among Pontryagin numbers of smooth closed manifolds. The torsion-free part of the oriented cobordism ring is a polynomial ring over  $\mathbb{Z}$  generated by a set of smooth closed manifolds in dimension  $4k$ 's:

$$\Omega_*^{SO}/\text{tor} \cong \mathbb{Z}[M^4, M^8, \dots]$$

where the generator  $M^{4k}$  can be taken as any manifold satisfying the following characteristic number property [St1]:

$$s_k(p_1, \dots, p_k)[M^{4k}] = \begin{cases} \pm q & \text{if } 2k+1 \text{ is a power of the prime } q; \\ \pm 1 & \text{if } 2k+1 \text{ is not a prime power} \end{cases}$$

Pontryagin numbers are oriented cobordism invariants. If we can find a set of basis manifold of  $\Omega_{4k}^{SO}/\text{tor}$  and compute the Pontryagin numbers, the congruence relations are then computable from the integer sublattice. Since  $s_k[\mathbb{CP}^{2k}] = 2k+1$ , in many of the  $4k$  dimensions (when  $2k+1 = q$  with  $q$  a prime),  $\mathbb{CP}^{2k}$  qualifies as a generator. For example, in dimension 8,

$$\Omega_8^{SO} \cong \langle \mathbb{CP}^2 \times \mathbb{CP}^2 \rangle \oplus \langle \mathbb{CP}^4 \rangle$$

Then for any smooth closed 8-dimensional manifold  $N$ , the Pontryagin number of  $N$  can be written as a linear combination

$$\begin{cases} p_{11}[N] = kp_{1,1}[\mathbb{CP}^2 \times \mathbb{CP}^2] + \ell p_{1,1}[\mathbb{CP}^4] = 18k + 25\ell \\ p_2[N] = kp_2[\mathbb{CP}^2 \times \mathbb{CP}^2] + \ell p_2[\mathbb{CP}^4] = 9k + 10\ell \end{cases}$$

with  $k, \ell \in \mathbb{Z}$ . Then the congruence relations among Pontryagin numbers of any 8-dimensional smooth closed manifold  $N$  can be computed as

$$\begin{cases} 5 \mid p_{1,1}[N] - 2p_2[N] \\ 9 \mid 2p_{1,1}[N] - 5p_2[N] \\ p_{1,1}[N] \in \mathbb{Z} \\ p_2[N] \in \mathbb{Z} \end{cases}$$

However, in dimensions such as  $4k = 16$  and  $4k = 28$  where  $2k+1$  is not a prime,  $\mathbb{CP}^{2k}$  does not satisfy the characteristic number property, thus fails to qualify as a generator. We have to construct a generating manifold from a disjoint union of  $\mathbb{CP}^{2k}$  and certain complex hypersurfaces [M1]. For example, in dimension  $4k = 16$ , we have

$$s_4(p)[9\mathbb{CP}^8 + \mathcal{H}_{3,6}] = -3$$

and in dimension  $4k = 28$

$$s_7(p)[-85\mathbb{CP}^{14} - 16\mathcal{H}_{3,12} + 2\mathcal{H}_{5,10}] = -1$$

where  $\mathcal{H}_{m,n}$  is the hypersurface of degree  $(1, 1)$  in  $\mathbb{CP}^m \times \mathbb{CP}^n$ . Once we obtain the generating manifolds, we still need to compute all the Pontryagin numbers  $p_I$  for a set of basis manifolds, which is a very tedious computation.

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ROSE-HULMAN INSTITUTE OF TECHNOLOGY, INDIANA, 47803  
*E-mail address:* su@rose-hulman.edu